Equivalence of tensor categories in 2d rational conformal field theory.

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- A conformal net $\mathcal{A}$ (Haag–Kastler net on S^1)

- A (unitary) vertex operator algebra (VOA) V.

$\mathcal{A}$ or V is the algebra of chiral fields
(chiral algebras)

- Chiral fields $\xrightarrow{\text{Tensor Categories}}$ Full fields

(real/complex 1d) (real 2d)

Task: Compare the two ways of constructing 2d full CFT.

Step 1. Relate $\mathcal{V}$ and $\mathcal{A}$.

Option 1	Option 2.
Smeared fields	Segal CFT
(Carpi - Kawahigashi - Longo - Weiner 18') CKLW	(Tener 19')

VOA 101

- a vector space V
- a collection of holomorphic fields on $z \in \mathbb{C} - \{0\}$

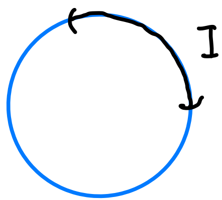
  $\varphi(z) \curvearrowright V$.

- We write $\varphi(z) = Y(v, z)$ where $v = \lim_{z \to 0} \varphi(z)\Omega$

Vacuum vector

CKLW's approach $V \rightsquigarrow \mathcal{A}_V$

- $\mathcal{A}_V \curvearrowright \mathcal{H}$ where $\mathcal{H}$ is the completion of V.

- 

  For $f \in C_c^\infty(I)$, the smeared operator

  $$Y(v,f) = \oint_{S^1} Y(v,z) f(z) \frac{dz}{2\pi i}$$

  is preclosed.

- $\mathcal{A}_V(I) = vN\{\overline{Y(v,f)} : v \in V,\ f \in C_c^\infty(I)\}$

Two crucial conditions on V

- **Polynomial energy bounds**

$Y(v,f)$ is bounded by $(1+L_0)^n$ for some $n \in \mathbb{N}$.

($L_0 \geq 0$ is the Hamiltonian)

- **Strong locality of vertex operators**

If $f \in C_c^\infty(I)$ and $g \in C_c^\infty(J)$ where $I \cap J = \phi$, then

$Y(v,f)$ and $Y(u,g)$ commute strongly, i.e.,

$$[vN\{\overline{Y(v,f)}\}, vN\{\overline{Y(v,g)}\}] = \phi$$

Let $\mathrm{Rep}(V)$ and $\mathrm{Rep}(\mathcal{A}_V)$ be the braided tensor categories of semisimple representations of V and $\mathcal{A}_V$.

Step 2. Find natural conditions on V (or $\mathcal{A}_V$) so that we can prove $\mathrm{Rep}(V) \cong \mathrm{Rep}(\mathcal{A}_V)$ in a systematic way.

Answer: Generalize the conditions on vertex operators (in CKLW's approach) to intertwining operators.

Intertwining Operators (I. O.) of V

- Let W_1, W_2, W_3 be V-modules. If $\mathcal{Y}$ is a type $\binom{W_3}{W_1\,W_2}$ I. O. then

  $w_1 \in W_1 \longmapsto \mathcal{Y}(w_1, z)$ is linear operator from W_2 to W_3 

- $\mathcal{Y}$ "intertwines" the actions of V on W_1, W_2, W_3.

- $v \mapsto Y(v, z)$ is a type $\binom{V}{V\,V}$ I. O.

- $N_{W_1 W_2}^{W_3} = \dim \left(\text{Type} \binom{W_3}{W_1\ W_2} \text{ I.O.} \right)$

- Let $\mathbb{W}_{\text{full}}$ be the "state space" of full CFT

  V: Chiral algebra V': Anti-chiral algera.

Full field operator $\xi \in \mathbb{W}_{\text{full}} \longmapsto Y_{\text{full}}(\xi; z, \bar{z}) \curvearrowright \mathbb{W}_{\text{full}}$

Then $Y_{\text{full}}(\cdot\,; z, \bar{z}) = \sum_{\mathcal{Y}, \mathcal{Y}'} \mathcal{Y}(\cdot, z) \otimes \mathcal{Y}'(\cdot, \bar{z})$

where $\mathcal{Y}$: an I.O. of V

$\mathcal{Y}'$: an I.O. of V'

Thm. (G. 20')

regular $\Rightarrow$ Rep(V) is modular (Huang 08')

Let V be a unitary, simple, regular VOA. Assume the following conditions hold. Then Rep(V) is equivalent to a unitary braided tensor subcategory of $\operatorname{Rep}(\mathcal{A}_V)$. $\left(\operatorname{Rep}(V) \lesssim \operatorname{Rep}(\mathcal{A}_V)\right)$

Moreover, if we can show $\#\operatorname{Irr}(\operatorname{Rep}(V)) \geq \#\operatorname{Irr}(\operatorname{Rep}(\mathcal{A}_V))$

then $\operatorname{Rep}(V) \simeq \operatorname{Rep}(\mathcal{A}_V)$.

Thm (Continued): These Conditions are

$\boxed{A}$ The vertex operators of V satisfy polynomial energy bounds and strong locality. (CKLW's conditions)

$\boxed{B}$ Any irreducible V-module admits a unitary structure.

Thm. (Continued)

$\boxed{C}$ Sufficiently many I.O. of V satisfy

① Polynomial energy bounds

② Strong intertwing property:

Assume $\mathcal{Y}$ is of type $\binom{W_3}{W_1\ W_2}$, then $\forall\, w_1 \in W_1$,

$f \in C_c^\infty(I)$, $g \in C_c^\infty(J)$ where $I \cap J = \phi$,

$Y(v,f)$ and $\mathcal{Y}(w_1, g)$ Commute strongly.

$$\begin{tikzcd} W_2 \arrow[r, "Y(v,f)"] \arrow[d, "\mathcal{Y}(w_1,g)"'] & W_2 \arrow[d, "\mathcal{Y}(w_1,g)"] \\ W_3 \arrow[r, "Y(v,f)"] & W_3 \end{tikzcd}$$

The following unitary regular VOAs satisfy

- Conditions $\boxed{A}$ $\boxed{B}$ $\boxed{C}$ (a method of Wassermann 98')
- $\# \mathrm{Irr}(\mathrm{Rep}(V)) \geq \# \mathrm{Irr}(\mathrm{Rep}(A_V))$ (Henriques 19')

They are:

△ WZW-models △ Lattice VOAs △ Their tensor products

△ Their regular cosets, including:

- Discrete series W-algebras of type ADE (in particular minimal models)
- Parafermion VOAs

Problem: What about V^G where $G \leq \mathrm{Aut}(V)$ is finite and V^G is one of the above examples?