Existence and uniqueness of solutions of the semiclassical Einstein equation in cosmological models

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Introduction

QFT vs. GR

How to describe quantum matter and gravity **interplay** ($\mathtt{Quantum\ gravity}$)?

In first approximation,

- **QFT on curved spacetimes**: quantum matter field ϕ on a physical state ω propagating over fixed Lorentzian manifolds $(\mathcal{M}, g)$
- **Semiclassical gravity**: backreaction on the background geometry

$$G_{\mu\nu} = 8\pi G \left\langle :T_{\mu\nu}: \right\rangle_\omega \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

- **Physical applications**: black hole physics (*Hawking radiation, evaporation*), cosmological models for the early Universe (*Starobinsky inflation*)

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Initial-value problem

A well-posed initial-value formulation of semiclassical gravity is requested to search for physically relevant solutions

Quantization

The quantum stress-energy tensor

How to compute $\langle :T_{\mu\nu}: \rangle_\omega$

- **Stress-energy tensor of a scalar field**

$$T_{\mu\nu} = \frac{1}{2}\nabla_\mu\nabla_\nu\phi^2 + \frac{1}{4}g_{\mu\nu}\Box\phi^2 - \phi\nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\phi\nabla_\rho\nabla_\sigma\phi +$$
$$+ \xi\left(G_{\mu\nu} - \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\phi^2 - \frac{1}{2}g_{\mu\nu}m^2\phi^2$$

- **Quantum scalar field** $\phi(f)$ over globally hyperbolic spacetimes $(\mathcal{M}, g)$:

$$\phi(Pf) = 0, \qquad P = -\Box + m^2 + \xi R, \qquad [\phi(f_1), \phi(f_2)] = i\Delta(f_1, f_2)\mathbb{1}$$

- **Hadamard point-splitting regularization** $:\phi^2:$, $:\phi\nabla_\mu\nabla_\nu\phi:$, $:T_{\mu\nu}:$

$$\omega_2 \doteq \mathcal{H}_{0^+} + \mathcal{W} \qquad \mathcal{H}_{0^+} = \lim_{\epsilon\to 0^+}\left(\frac{U}{\sigma_\varepsilon} + V\log\left(\frac{\sigma_\varepsilon}{\lambda^2}\right)\right) \qquad \textit{Hadamard state}$$

- **Renormalization freedoms**

$$:\tilde{T}_{\mu\nu}: = :T_{\mu\nu}: + c_1 m^4 g_{\mu\nu} + c_2 m^2 G_{\mu\nu} + c_3 I_{\mu\nu} + c_4 J_{\mu\nu}$$

- **Locality and conservation**

$$\langle :T_{\mu\nu}: \rangle_\omega \doteq \lim_{x'\to x} D_{\mu\nu}\left(\omega_2(x,x') - H_{0^+}(x,x')\right) \quad \textit{locally covariant}$$

$$\nabla^\mu \langle :T_{\mu\nu}: \rangle_\omega = 0 \quad \textit{covariantly conserved}$$

Semiclassical Einstein Equation on cosmological spacetimes

Semiclassical Einstein equations in cosmological models (1/2)

Cosmological spacetime

- **Friedmann-Lemaître-Robertson-Walker metric** $(\mathcal{M}, g)$, where $\mathcal{M} = I_t \times \Sigma$

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a(t)^2 \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i$$

  - t **cosmological time**
  - $a(t)$ **scale factor** describes the history of the Universe
  - $H(t) = \frac{1}{a}\frac{da}{dt}$ the **Hubble function**
  - $d\tau = a^{-1}dt$ **conformal time** such that $g = a^2\eta$, $\eta = (-1, 1, 1, 1)$

Classical model for the matter

- **Matter as perfect fluid** $T_\mu{}^\nu = \mathrm{diag}(-\varrho, p, p, p)$
- **Equation of state** $p = w\varrho$, where $w = 0, 1/3, -1$ for ordinary matter, radiation and dark energy due to a cosmological constant Λ.
- **Friedmann equations**

$$H^2 = \frac{8\pi G}{3}\varrho + \frac{\Lambda}{3}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varrho + 3p) + \frac{\Lambda}{3}$$

From now, $\Lambda = 0$.

Semiclassical Einstein equations in cosmological models (2/2)

Semiclassical cosmological model

- **Semiclassical equations**

$$-R = 8\pi G \langle :T: \rangle_\omega \qquad \nabla^\mu \langle :T_{\mu\nu}: \rangle_\omega = 0$$

- The initial condition is the validity of the **energy constraint** at the initial time

$$H_0^2 = \frac{8\pi G}{3} \langle :\varrho_0: \rangle_\omega$$

- Since the state is not a local object, the semiclassical equations are **non-local**
- **Initial-value formulation** for the FLRW spacetime $(\mathcal{M}, g)$ and the free quantum matter field (ϕ, ω) with initial data $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$

$$\begin{cases} -R = 8\pi G \langle :T: \rangle_\omega \\ G_{00}(\tau_0) = 8\pi G \langle :T_{00}: \rangle_\omega (\tau_0) \\ \nabla^\mu \langle :T_{\mu\nu}: \rangle_\omega = 0 \quad \checkmark \end{cases}$$

- We look for **existence and uniqueness** of solutions of that system in case of non-conformal coupling, namely for $\xi \neq \frac{1}{6}$.

Energy constraint

Quantum state for FLRW spacetimes (1/3)

First step

Fix a regular state ω at $\tau = \tau_0$ which fulfils the initial energy constraint

$$G_{00}(\tau_0) = 8\pi G \left\langle :T_{00}: \right\rangle_\omega (\tau_0)$$

- **Vacuum-like state**: quasi-free, pure, homogeneous and isotropic

$$\omega_2(x,y) = \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\bar{\zeta}_k(\tau_x)}{a(\tau_x)}\frac{\zeta_k(\tau_y)}{a(\tau_y)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-\epsilon k} d\vec{k} \qquad k \doteq |\vec{k}|$$

- **Temporal modes** $\zeta_k(\tau)$

$$\zeta_k''(\tau) + \Omega_k^2(\tau)\zeta_k(\tau) = 0 \qquad \Omega_k = \sqrt{k^2 + w(\tau)}$$

  with the potential $w(\tau) = a^2 m^2 + (\xi - 1/6)a^2 R > 0$.

- **Conformal vacuum state** ω^c: a solution $\chi_k(\tau)$ can be constructed as convergent Dyson series using a perturbative potential $V(\tau) = w(\tau) - w(\tau_0)$ with respect to the initial conditions

$$\chi_k(\tau_0) = \frac{1}{\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}k_0\tau_0} \qquad \chi_k'(\tau_0) = \frac{\mathrm{i}k_0}{\sqrt{2k_0}} \mathrm{e}^{\mathrm{i}k_0\tau_0} \qquad k_0^2 \doteq \Omega_k^2(\tau = \tau_0).$$

Quantum state for FLRW spacetimes (2/3)

- **Point-splitting regularization mode-wise** for $\left\langle :\phi^2: \right\rangle_\omega$ and $\langle :T_{00}: \rangle_\omega$

$$\left\langle :\phi^2: \right\rangle_\omega = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)^3 a^2} \int_{\mathbb{R}^3} \left(|\zeta_k|^2 - C^{\mathcal{H}}_{\phi^2}(\tau, k) \right) e^{-\epsilon k} \mathrm{d}\vec{k} + \ \mathit{rest}$$

$$\langle :T_{00}: \rangle_\omega = \frac{1}{(2\pi)^3 a^4} \int_{\mathbb{R}^3} \left(\varrho_k[\zeta_k, \zeta'_k] - C^{\mathcal{H}}_{\varrho}(\tau, k) \right) \mathrm{d}\vec{k} + \ \mathit{rest}$$

  Functions $C^{\mathcal{H}}_{\phi^2}, C^{\mathcal{H}}_{\varrho}$ and rests depend up to $a^{(3)}(\tau)$ ▸ point-splitting

- ω^c gives a finite $\left\langle :\phi^2: \right\rangle_{\omega_c} \in C^0([\tau_0, \tau_1])$, but it **is not regular** enough for $\xi \neq 1/6$ to compute $\partial_\tau \left\langle :\phi^2: \right\rangle_\omega$ and $\langle :T_{00}: \rangle_\omega$
- Changing the initial conditions of χ_k corresponds to change the state with a **Bogoliubov transformation**

$$\zeta_k(\tau) = A_k \chi_k(\tau) + B_k \bar{\chi}_k(\tau)$$

- Coefficients A_k, B_k are fixed by the initial data and can be chosen sufficiently regular to construct other states

Regular states

- Modes ζ_k define a **regular state** ω if at $\tau = \tau_0$

$$|\zeta_k^2(\tau)| - C^{\mathcal{H}}_{\phi^2}(\tau, k), \quad \frac{d}{d\tau}\left[|\zeta_k^2|(\tau) - C^{\mathcal{H}}_{\phi^2}(\tau, k)\right], \quad \varrho_k[\zeta_k, \zeta_k'] - C^{\mathcal{H}}_{\varrho}(\tau, k) \in L^1(k^2 dk)$$

  and

$$\left\langle :\phi^2: \right\rangle_\omega \in C^2\left([\tau_0, \tau_1]\right) \qquad \langle :T_{00}: \rangle_\omega \in C^0\left([\tau_0, \tau_1]\right)$$

- **Examples**: Parker's fourth-order adiabatic states (*Phys.Rev.D 36, 2963*), instantaneous vacuum state (*Phys.Rev.D 91, 064051*), Olbermann states of low energy (*Class. Quantum Grav. 24*), ...

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Proposition

Given the initial data $\left(a_0, a_0', a_0'', a_0^{(3)}\right)$*, it is always possible to select initial conditions that fix the regular state* ω *in such a way that the* **energy constraint**

$$G_{00} = 8\pi G \langle :T_{00}: \rangle_\omega$$

is satisfied at $\tau = \tau_0$*.*

From now, the quantum state ω will be considered fixed once and for all

The traced semiclassical Einstein equation

Semiclassical Einstein equations in cosmological models (3/3)

Second step

Solving the **traced semiclassical Einstein equation**: $-R = 8\pi G \langle{:}T{:}\rangle_\omega$

$$\langle{:}T{:}\rangle_\omega = \left(3\left(\xi - \frac{1}{6}\right)\Box - m^2\right)\left\langle{:}\phi^2{:}\right\rangle_\omega + T_A + \beta_1 m^4 + \beta_2 m^2 R + \beta_3 \Box R.$$

- **State-dependent contribution**:

$$\left\langle{:}\phi^2{:}\right\rangle_\omega = \lim_{x'\to x}\left(\omega_2(x,x') - \mathcal{H}_{0^+}(x,x')\right) = \mathcal{W}(x,x)$$

- **Trace anomaly** arising because $\mathcal{H}$ is not solution of the equation of motion:

$$T_A = \frac{1}{4\pi^2}\left(\frac{(6\xi-1)^2R^2}{288} + \frac{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}}{720}\right) \qquad \textit{up to ren. fr.}$$

- **Renormalization constants**:

$$\beta_1 \Rightarrow \Lambda \qquad \beta_2 \Rightarrow G \qquad \beta_3 \Rightarrow \textit{Quantum freedom}$$

- **Non-classical dynamics**:
  1. $\Box\left\langle{:}\phi^2{:}\right\rangle_\omega$ and $\Box R$ contain **higher order derivatives** of $a(\tau)$ than the second
  2. $\Box\left\langle{:}\phi^2{:}\right\rangle_\omega$ is highly **non-local functional** of $a(\tau)$

Partial integration

- We shall rewrite this semiclassical equation as a system of equations

$$\begin{cases} (-\Box + M_c)F = S \\ \langle :\phi^2: \rangle_\omega - c_\xi R = F \end{cases}$$

where $M_c \propto m^2$, $c_\xi \propto \beta_3$ and the source function

$$S \doteq \frac{1}{3(\xi - 1/6)} \left(\beta_1 m^4 + \frac{R}{8\pi G} + \beta_2 m^2 R + \beta_3 M_c R + T_A \right)$$

depends on the derivatives of $a(\tau)$ up to the **second order**

- In case of FLRW,

$$\begin{cases} P_c F = S \\ \langle :\phi^2: \rangle_\omega - c_\xi R = F \end{cases} \qquad P_c \doteq \frac{1}{a^3(\tau)} \left(\partial_\tau^2 + a^2(\tau) M_c - \frac{1}{6} a^2(\tau) R \right) a(\tau)$$

- The initial data $(F, F')(\tau_0) = (F_0, F'_0)$ for the function $F(a, R)$ are constructed out to the initial data of the geometry $\left(a_0, a'_0, a''_0, a_0^{(3)}\right)$ and the regular state ω
- There exist a **unique solution** $F(a, r)$ in the finite interval $[\tau_0, \tau_1]$ which depends continuously on the initial data.
- Estimates for the solution can be obtained using the **Grönwall lemma** or constructing the **retarded operator** Δ_c^R related to P_c

New semiclassical equation

- The traced semiclassical Einstein equation is reduced to

$$\left\langle :\phi^2: \right\rangle_\omega = \mathfrak{S} \qquad \mathfrak{S} \doteq c_\xi R + F(a, R)$$

  where $F(a, R)$ is the unique solution constructed previously
- We shall study its **time derivative**

$$\partial_\tau \left(a^2 \left(\left\langle :\phi^2: \right\rangle_\omega - c_\xi R - F(a, R) \right) \right) = 0.$$

  in order to be able to impose the initial data $a^{(3)}(\tau_0) = a_0^{(3)}$
- Prove that $\partial_\tau \left(a^2 \left(\left\langle :\phi^2: \right\rangle_\omega - c_\xi R - F(a, R) \right) \right) = 0$ is a **fixed point equation**

$$X' = \mathcal{C}[X'] \qquad X \doteq \frac{1}{6} a^2 R = \frac{a''}{a}$$

- Construct a **contraction map** $\mathcal{C} : \mathcal{B}_\delta \subset C([\tau_0, \tau_1]) \to \mathcal{B}_\delta$ on the closed ball

$$\mathcal{B}_\delta \doteq \left\{ X' \in C([\tau_0, \tau_1]) \mid X'(\tau_0) = X'_0 \right\} \qquad \delta > 0$$

  equipped with the uniform norm $\| \cdot \|_\infty$, when $\tau_1 - \tau_0$ is sufficiently small
- The existence and uniqueness of a solution is consequence of the **Banach fixed point theorem**

The unbounded operator in the state contribution

The source of the regularity issues

- The analysis of $\partial_\tau \left(a^2 \langle :\phi^2: \rangle_\omega \right)$ yields that the semiclassical equation has the form

$$\mathcal{T}_{\tau_0}[V'] = h \qquad V = m^2\left(a^2 - a_0^2\right) + (\xi - 1/6)\left(\frac{a''}{a} - \frac{a_0''}{a_0}\right).$$

where h is a combination of functions and functionals of V

$$\mathcal{T}_{\tau_0}[f] \doteq -\frac{1}{8\pi^2}\int_{\tau_0}^{\tau} f'(\eta)\log(\tau-\eta)\mathrm{d}\eta \qquad f \in \mathcal{C}^1\left([\tau_0,\tau]\right).$$

1. **Retarded operator**: $\mathcal{T}_{\tau_0}$ depends on $[\tau_0, \tau]$
2. **Higher-order derivative**: $\mathcal{T}_{\tau_0}$ acts on V'', which contains $a^{(4)}$.
3. **Unbounded operator**: one can prove that $\|\mathcal{T}_{\tau_0}[f]\|_\infty \not\leq C\|f\|_\infty$
4. **Generality**: $\mathcal{T}_{\tau_0}$ does not depend on the details of the state

- A way to overcome this problem is to study an **inversion formula** for $\mathcal{T}_{\tau_0}[f] = h$ and to prove the continuity of the associated inverse operator $\mathcal{T}_{\tau_0}^{-1}[h]$

The inverse operator

Properties of $\mathcal{T}_{\tau_0}^{-1}$

- The inversion formula for $h = \mathcal{T}_{\tau_0}[f]$, $f \in C^1([\tau_0, \tau_1])$ is

$$f(\tau) = f(\tau_0) + \int_{\tau_0}^{\tau} K(\tau - \eta) h(\eta) \mathrm{d}\eta$$

where the kernel K is obtained by the **inverse Laplace transform** and yields

$$K(x) \doteq \frac{1}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{e}^{sx} \frac{8\pi^2}{\gamma + \log s} \mathrm{d}s, \qquad \alpha > \mathrm{e}^{-\gamma}, \qquad \gamma : \textit{Euler-Mascheroni constant}$$

- The inverse operator $\mathcal{T}_{\tau_0}^{-1}[h]$ is continuous on $C([\tau_0, \tau])$:

$$\|\mathcal{T}_{\tau_0}^{-1}[h]\|_\infty \leq \left(\int_{\tau_0}^{\tau} |K(\tau - \eta)| \mathrm{d}\eta \right) \|h\|_\infty \leq C_\infty \|f\|_\infty.$$

- The constant C_∞ depends continuously on $\tau - \tau_0$ and **vanishes for** $\tau \to \tau_0$
- It is a **retarded operator**, so causality is respected

$\mathcal{T}_{\tau_0}^{-1}$ can play the role of **contraction map** in the semiclassical equation: adopting the inversion formula, equation $\mathcal{T}_{\tau_0}[V'] = h$ can be written as

$$V' = V_0' + \mathcal{T}_{\tau_0}^{-1}[h]$$

Existence and uniqueness of solutions of local mild solutions (1/2)

The semiclassical equation becomes a fixed point equation

$$X' = \mathcal{C}[X'] \qquad \mathcal{C}[X'] \doteq X_0' - \frac{2m^2}{(6\xi - 1)}(aa' - a_0 a_0') - \frac{1}{(6\xi - 1)}\mathcal{T}_{\tau_0}^{-1}[h]$$

Proposition

Fix $\delta > 0$ and let $\mathcal{B}_\delta$ the closed ball in the Banach space $C([\tau_0, \tau_1])$ with finite $\tau_1 > \tau_0$, centred in X_0'. For τ_1 sufficiently small, the map $\mathcal{C} : \mathcal{B}_\delta \subset C([\tau_0, \tau_1]) \to \mathcal{B}_\delta$ is a **contraction** *on $\mathcal{B}_\delta$, namely there exists $C \in (0, 1)$ such that*

$$\|\mathcal{C}[X'] - X_0'\|_\infty \leq \delta, \qquad \|\mathcal{C}[X_2'] - \mathcal{C}[X_1']\|_\infty = C\|X_2' - X_1'\|_\infty$$

Hence, there exists a **unique fixed point** *of the equation $X' = \mathcal{C}[X']$ in $\mathcal{B}_\delta$.* ▸ details

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$$X' = \mathcal{C}[X'] \qquad \mathcal{C}[X'] \doteq X'_0 - \frac{2m^2}{(6\xi - 1)}(aa' - a_0 a'_0) - \frac{1}{(6\xi - 1)}\mathcal{T}^{-1}_{\tau_0}[h]$$

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Hence, there exists a **unique fixed point** *of the equation $X' = \mathcal{C}[X']$ in $\mathcal{B}_\delta$.* details

Theorem

Given some initial data $(a_0, a'_0, a''_0, a_0^{(3)})$ at $\tau = \tau_0$ and a regular quasi-free compatible state ω, there exists a non-empty interval $[\tau_0, \tau_1]$ and a closed ball $\mathcal{B}_\delta$ such that a **unique mild solution** of $X' = \mathcal{C}[X']$ exists

Conclusions

Summary and final comments

- Solving the semiclassical Einstein equation in FLRW spacetimes means to apply the **Banach fixed-point theorem** and construct a contraction map
- When higher-order derivatives terms are involved, the proof is achieved after rewriting the semiclassical equation in a **new non-standard form**
- Looking for **numerical algorithms** to find approximate solutions

Open questions

- Existence and uniqueness of **strong solutions** and **global solutions**
- Implications on cosmological models (inflation)
- Can this analysis be applied to **other spacetimes**?

References

- *Utiyama & DeWitt (1962)*, *Starobinski (1980)*, *Anderson (1983)*
- *Pinamonti (2011)*, *Pinamonti & Siemssen (2015)*
- *Gottschalk & Siemssen (2018)*

Thanks for the attention!

Expectation values of $:\phi^2:$ and $:T_{00}:$

Point-splitting regularization mode-wise

$$\langle :\phi^2: \rangle_\omega = \frac{1}{(2\pi)^3 a^2} \int_{\mathbb{R}^3} \left(|\zeta_k|^2 - C^{\mathcal{H}}_{\phi^2}(\tau, k) \right) \mathrm{d}\vec{k} + \frac{w(\tau)^2}{8\pi^2 a^2} \log\left(\frac{w(\tau_0)}{a(\tau)} \right) - \frac{w(\tau_0)^2}{16\pi^2 a^2} + \alpha_1 m^2 + \alpha_2 R(\tau)$$

$$\langle :T_{00}: \rangle_\omega = \frac{1}{(2\pi)^3 a^4} \int_{\mathbb{R}^3} \left(\frac{|\zeta_k'|^2}{2} + \left(k^2 + a^2 m^2 - (6\xi - 1)\, a^2 H^2 \right) \frac{|\zeta_k|^2}{2} + aH\,(6\xi - 1)\, 2\mathrm{Re}(\overline{\zeta}_k \zeta_k') - C^{\mathcal{H}}_{\varrho}(\tau, k) \right) \mathrm{d}\vec{k}$$
$$- \frac{H^4}{960\pi^2} + \left(\xi - \frac{1}{6} \right)^2 \frac{3H^2 R}{8\pi^2} + k_1 m^4 + k_2 m^2 G_{00} + k_3 I_{00}$$

Point-splitting functions

$$C^{\mathcal{H}}_{\phi^2}(\tau, k) \doteq \frac{1}{2k_0} - \frac{V(\tau)}{4k_0^3},$$

$$C^{\mathcal{H}}_{\varrho}(\tau, k) \doteq \frac{k}{2} + \frac{a^2 m^2 - a^2 H^2 (6\xi - 1)}{4k} - \frac{a^4 m^4 + 12\left(\xi - \frac{1}{6}\right) m^2 a^4 H^2 - a^4 \left(\xi - \frac{1}{6}\right)^2 2 I_{00}(\tau)}{16k(k^2 + \frac{a^2}{\lambda^2})}$$

References

J. Schlemmer (*PhD Thesis*), A. Degner (*PhD Thesis*), T.P. Hack (*arXiv:1306.3074s*), D. Siemssen (*arXiv:1503.01826*)

Details about the contraction map $\mathcal{C}$

Functional derivative

- Given a functional $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ in a Banach space $\mathcal{D}$, $\mathcal{F}$ is **Gateaux differentiable** at $V \in \mathcal{D}$ if the exists the **functional (or Gateaux) derivative**

$$\delta\mathcal{F}[V, W] \doteq \lim_{\epsilon \to 0^+} \frac{\mathcal{F}[V + \epsilon W] - \mathcal{F}[V]}{\epsilon} \qquad \forall\, W \in \mathcal{D}$$

- If $\delta\mathcal{F}$ is (uniformly) continuous in V for every W, then $\mathcal{F}$ is **locally Lipschitz**

$$\|\delta\mathcal{F}[V, W]\|_{\mathcal{D}} \leq C\|W\|_{\mathcal{D}} \quad \Rightarrow \quad \|\mathcal{F}[V] - \mathcal{F}[W]\|_{\mathcal{D}} \leq C\|V - W\|_{\mathcal{D}}$$

Strategy of the proof

- $\mathcal{C}$ is a linear combination of compositions of functions or functionals of a, V and X which are **continuous** and have continuous functional derivative with respect to X.
- Fixed the initial data, X is **uniquely** assigned from X' by

$$X(\tau) = X_0 + \int_{\tau_0}^{\tau} X'(\eta)\mathrm{d}\eta$$

  and determines a unique **FLRW spacetime** $(\mathcal{M}, g[X])$ with the scale factor $a[X](\tau)$ constructed as the unique solution of $a'' = Xa$
- The proof follows from the continuity of $\mathcal{T}_{\tau_0}^{-1}$ and from the property of C_∞