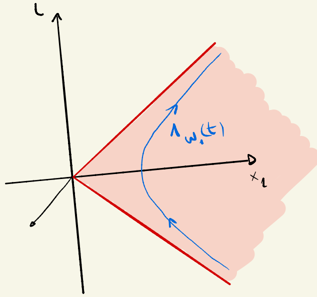
Covariant Homogeneous Nets of Standard Subspaces



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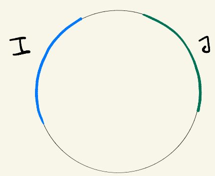
1st Virtual LQP Workshop



joint ongoing work with K.-H. Neeb

18th June 2020

- Locality Vs Symmetry in QFT

- The **Bisognano–Wichmann** property identify geometric and algebraic objects in a AQFT.

- The algebraic construction of the free fields starts from the geometric data to define local algebras ($\mathbb{R}^{1,s}$, S^1, dS^n)

- Three moments in this construction
  - * Identification: Boost – Wedge regions
  - * Representation Theory ⇒ First Quantization
  - * Second Quantization ⇒ AQFT

- Given a Lie group G (no spacetime needed) we investigate a possible definition of a QFT starting from an

Abstract notion of Wedge Regions

OUTLINE

I. PRELIMINARIES

II. BRUNETTI-GUIDO-LONGO CONSTRUCTION

FREE FIELD

III. COVARIANT NETS OF STANDARD SUBSPACES

1. PRELIMINARIES

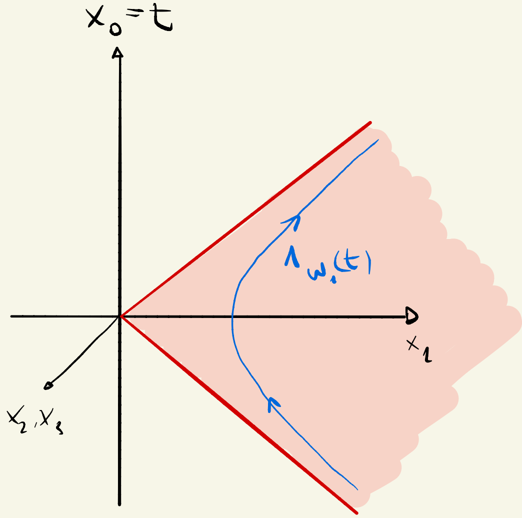
Preliminaries

▷ Features of wedge regions can be determined at Lie algebra level.

▷ Minkowski $(\mathbb{R}^{1+s}, g = \mathrm{diag}(1,-1,\dots,-1))$

Symmetry group: Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$

▷ Wedge regions are characterized by their symmetries



$$W_1 = \{ x \in \mathbb{R}^{1+3} \mid |x_0| < x_1 \}$$

$$\Lambda_{W_1}(t) = \begin{pmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t & 0 & 0 \\ \mathrm{sh}\, t & \mathrm{ch}\, t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{L}_+^\uparrow$$

$$\Lambda_{W_1}(t) W_1 = W_1$$

Wedge regions

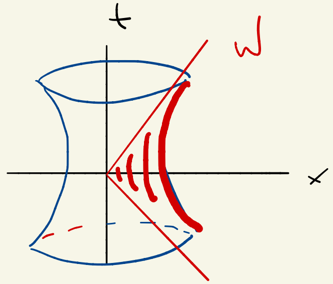
$$\mathcal{W}_+ \ni W = g W_1 \quad \xleftrightarrow[g \in \mathcal{P}_+^\uparrow]{1:1} \quad \Lambda_W = g \Lambda_{W_1} g^{-1} \in \mathcal{P}_+^\uparrow$$

One-parameter groups of boosts

PRELIMINARIES — FURTHER EXAMPLES

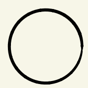
▷ $\mathcal{L}_+^\uparrow = PSL(2,\mathbb{R}) \curvearrowright dS^2 = \{ (t,x) \mid t^2 = x^2 - m^2 \}$

WEDGES :



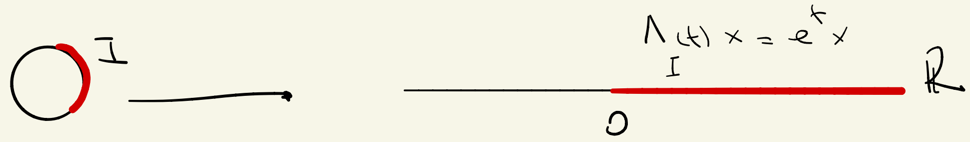
$\longleftrightarrow \Lambda_W$ BOOST

▷ $\text{Möb} = PSL(2,\mathbb{R}) \curvearrowright$



$S^1 \simeq \overline{\mathbb{R}}$ - CHIRALITY

WEDGES : INTERVALS $I \xleftrightarrow{1:1} \Lambda_I^{(t)}$ DILATIONS



▷ WE CAN DEFINE A COVARIANT SET OF WEDGES JUST STARTING FROM THE LIE ALGEBRA $\mathfrak{g}$ OF THE SYMMETRY GROUP G $(= \mathcal{P}_+^\uparrow, \mathcal{L}_+^\uparrow, \text{Möb}..)$

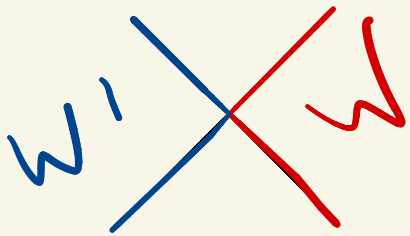
$$x_W \in \mathfrak{g} \xleftrightarrow{1:1} \Lambda_W(t) = e^{x_W t} \xleftrightarrow{1:1} W$$

$$x_{gW} = g\, x_W\, g^{-1} \longleftrightarrow gW$$

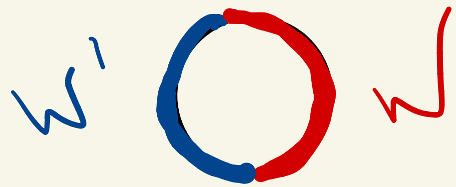
PRELIMINARIES

▷ THE **COMPLEMENT** WEDGE : $x_{W'} = -x_W \leftrightarrow \Lambda_{W'} = \Lambda_W^{-1} \leftrightarrow W'$

$\mathbb{R}^{1+s}$:



S^1 :

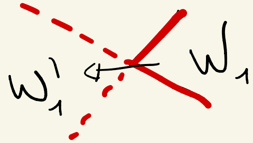


$\{x_W\}$ IS A LOCAL G-COVARIANT SET OF LIE GENERATORS

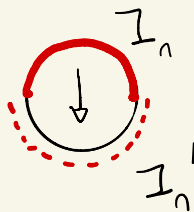
FURTHER SIMMETRIES

▷ POSITION-TIME REFLECTION (PT SYMMETRY)

MINKOWSKI SPACE & de Sitter SPACE : $j_{W_1}(t,\underline{x}) = (-t, -x_1, x_2, x_3)$



S^1 : $j_{I_n}^2 = \mathbb{1}$



$\subset \mathbb{C}$

THEN j_W IS DEFINED BY COVARIANCE

PROPERTIES

1* $j_W W = W'$

2* $j_W = \Lambda_W(i\pi)$

PRELIMINARIES

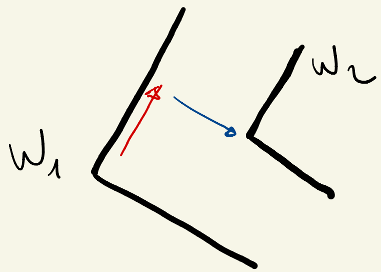
INCLUSION IF THERE EXISTS A POSITIVE CONE $C \subset \mathfrak{g}$

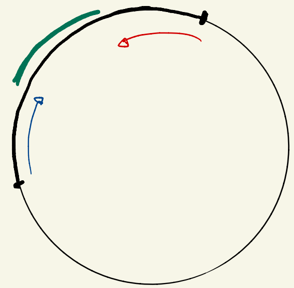
SUCH THAT $gCg^{-1} \subseteq C$, $g \in G$ (NEGATIVE CONE $-C$)

e.g. $V_+ \subset \mathbb{R}^{1+s}$, $C_+ \subset \mathfrak{sl}_2(\mathbb{R})$

THEN INCLUSIONS ARE OBTAINED BY COMPOSITION OF TRANSLATIONS:

$$e^{tP_\pm} \in G, \quad t \geq 0, \qquad P_\pm \in \pm C \quad :$$





WHERE $\pm P_\pm$ GENERATE WITH $\pm x_W$ TRANSLATION-DILATION GROUP

$$\mathrm{ad}\, x_W (P_\mp) = \pm P_\pm \iff P_\pm \in \ker(\mathrm{ad}\, x_W \mp 1)$$

11. BRUNETTI-GUIDO-LONGO CONSTRUCTION

FREE FIELD

STANDARD SUBSPACES

▷ $H \subset \mathcal{H}$ REAL CLOSED SUBSPACE IS **STANDARD** IF

$$\overline{H + iH} = \mathcal{H}\,, \quad H \cap iH = \{0\}$$

SYMPLECTIC COMPLEMENT $\quad H' = \{\xi \in \mathcal{H} : \operatorname{Im}(\xi, \eta) = 0\,,\ \forall\, \eta \in H\}$

TOMITA OPERATOR : $\quad S_H : H + iH \longrightarrow H + iH \quad$ CLOSED ANTILINEAR INVOLUTION

$$\xi + i\eta \longmapsto \xi - i\eta$$

POLAR DECOMPOSITION : $\quad S_H = J_H \Delta_H^{1/2}\,,\quad$ J ANTI-UNITARY MODULAR CONJUGATION

Δ SELF-ADJOINT MODULAR OPERATOR

Δ^{it} MODULAR GROUP

AND $\quad (\star)\ \boxed{J \Delta_H^{it} J = \Delta_H^{it}} \quad \boxed{J_H H = H'\,,} \quad \boxed{\Delta_H^{it} H = H\,,}$

$$S_H \xleftrightarrow{1-1} (J, \Delta) \text{ WITH } (\star) \xleftrightarrow{1-1} H \text{ STANDARD SUBSPACE}$$

BRUNETTI – GUIDO – LONGO (BGL) CONSTRUCTION

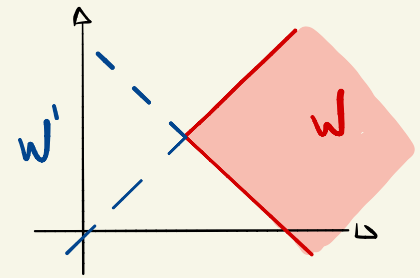
▷ $\mathcal{P}_+ = \mathcal{P}_+^\uparrow \cup \mathcal{P}_+^\downarrow$ PROPER POINCARÉ GROUP (also Möb, $\mathcal{L}_+^\uparrow$)

$$U \text{ (ANTI-) UNITARY} = \begin{cases} \text{UNITARY ON } \mathcal{P}_+^\uparrow \\ \text{ANTI-UNITARY ON } \mathcal{P}_+^\downarrow = j_W \cdot \mathcal{P}_+^\uparrow \end{cases}$$ POSITIVE ENERGY

$$H(W) : \longmapsto (J_W, \Delta_W) = \left(U(j_W), e^{-2\pi X_W} \right)$$ STANDARD SUBSPACE

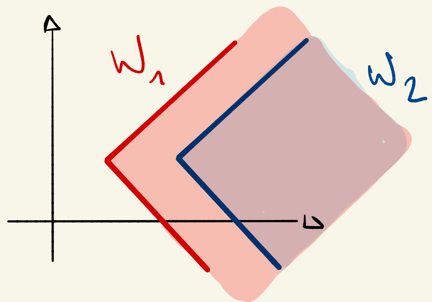
WHERE $U(\Lambda_W(t)) = e^{i X_W t}$

FIRST QUANTIZATION



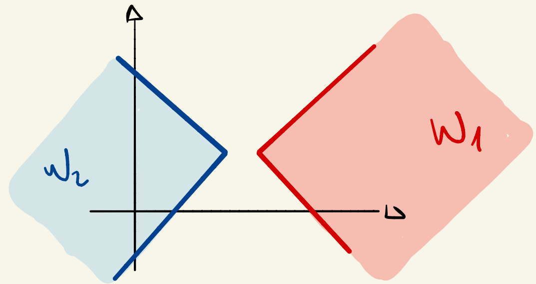
$W \longmapsto H(W) \subset \mathcal{H}$

ISOTONY



$W_2 \subset W_1 \Rightarrow H(W_2) \subset H(W_1)$

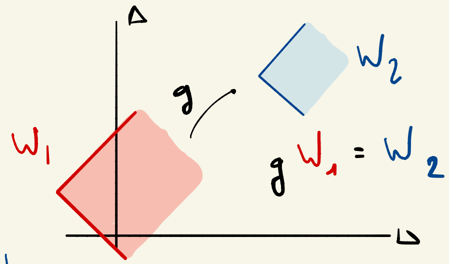
LOCALITY



$W_1 \subset W_2' \Rightarrow H(W_1) \subset H(W_2)'$

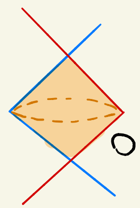
POINCARÉ COVARIANCE

$\mathcal{U} : \mathcal{P}_+^\uparrow \longrightarrow \mathcal{U}(\mathcal{H})$ st $\mathcal{U}(\mathbb{R}^4) \geq 0$



$\mathcal{U}(g) H(W_1) = H(W_2)$

THEN $$H(O) = \bigcap_{O \subset W} H(W)$$



BRUNETTI – GUIDO – LONGO (BGL) CONSTRUCTION

BISOGNANO WICHMANN PROPERTY : $U(\Lambda_W(2\pi t)) = e^{i2\pi X_W t} = \Delta_{H(W)}^{-it}$

PCT THEOREM : $U(j_W) = J_{H(W)}$

THEN SECOND QUANTIZATION

$$W \longmapsto A(W) = \{ w(h) : h \in H(W) \}'' \subset B(\mathcal{F}(H))$$

$$w(h)\, w(k) = e^{i\, \mathrm{Im}(h,k)}\, w(h+k), \quad h,k \in H$$ WEYL RELATIONS

$$O \longrightarrow A(O) = \Big(\bigcap_{W \supset O} A(W) \Big)''$$

SATISFY FUNDAMENTAL QUANTUM & RELATIVISTIC PRINCIPLES

⟹ ON A GENERAL GROUP G WE NEED TO DETERMINE A FIRST QUANTIZATION NET (SECOND IS A FUNCTOR)

III. COVARIANT NETS OF STANDARD SUBSPACES

LIE WEDGE

SETTING: G - GRADED LIE GROUP, $\varepsilon_G : G \to \{\pm 1\}$, $G^{\uparrow} = \varepsilon_G^{-1}(1)$

$$G = G^{\uparrow} \rtimes \mathbb{Z}_2 = G^{\uparrow} \rtimes \{e, \sigma\} = G^{\uparrow} \cup G^{\downarrow} = G^{\uparrow} \cup \sigma \cdot G^{\uparrow}$$

IN THIS TALK WE ASSUME $Z(G^{\uparrow}) = \{e\}$, CONNECTED

$\mathfrak{g}$ REAL LIE ALGEBRA OF $G^{\uparrow}$

ACTION: $\mathrm{Ad}^{\varepsilon} : G \to \mathrm{Aut}(\mathfrak{g})$, $\mathrm{Ad}^{\varepsilon} = \varepsilon_G(g)\,\mathrm{Ad}(g)$

DEF $\rhd$ $x \in \mathfrak{g}$ **EULER ELEMENT** IF $\mathrm{Sp}(\mathrm{ad}\,x) \subset \{-1, 0, 1\}$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$\mathfrak{g}_\nu = \mathrm{Ker}(\mathrm{ad}\,x - \nu \mathbf{1})$ W.R.T. ADJOINT REPRESENTATION $\quad \mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto [x, \cdot]$

$\rhd$ WE CAN ASSUME σ **EULER INVOLUTION** FOR x EULER ELEMENT

$$\mathrm{Ad}(\sigma) = e^{i\pi\,\mathrm{ad}\,x}$$

THEN BY COVARIANCE $\sigma_{gx} = g \cdot x$ IS EULER INVOLUTION FOR $g \cdot x$ EULER ELEMENT

EULER WEDGE

↳ DEF.

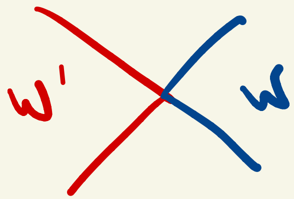
$$\boxed{W = (x, \sigma_x) \qquad \begin{array}{l} x \text{ EULER ELEMENT} \\ \sigma_x \text{ EULER INVOLUTION} \end{array}}$$

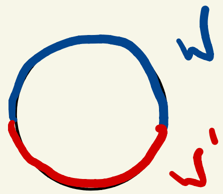
* G - ACTION : $\boxed{g.W = (Ad^{\varepsilon}(g)x,\ g\sigma g^{-1})}$

$\Rightarrow$ $e^{X_W t}.W = W$

* COMPLEMENT WEDGE $W' = (-x, \sigma)$

* $\sigma_x(X) = -X \Rightarrow \sigma W = W'$





Ex. $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $x = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ EULER ELEMENT

$\mathfrak{g} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{R}x \oplus \mathbb{R}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ 3-GRADING

$Ad\, \sigma_x = Ad\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ EULER INVOLUTION

$PSL_2(\mathbb{R})$ SYMMETRY GROUP FOR S^1 & dS^2:

SAME WEDGES, DIFFERENT RELATIONS DEFINED BY INCLUSIONS

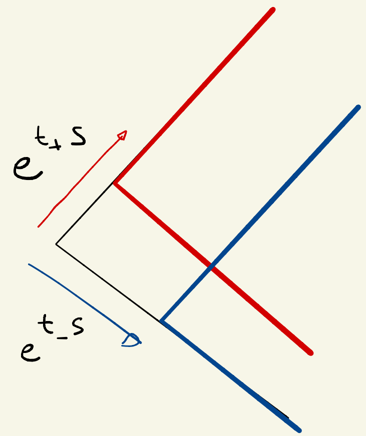
EULER WEDGES

$W = (x, \sigma_x)$ WEDGE

POSITIVE CONE: $C \subset \mathfrak{g}$ s.t. $\boxed{\mathrm{Ad}(g)\, C = \varepsilon_G(g)\, C \qquad g \in G}$

WEDGE ENDOMORPHISM SEMIGROUP

$$L_W = C_+ \oplus \mathfrak{g}_0 \oplus C_-$$



where $C_\pm = \pm C \cap \ker(\mathrm{ad}\, x \mp \mathrm{id})$, $e^{\mathfrak{g}_0} W = W$

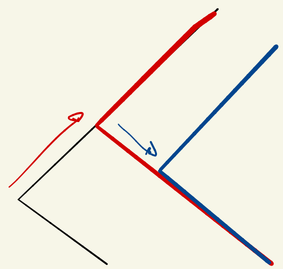
$$S_W = \exp C_+ \, \exp \mathfrak{g}_0 \, \exp C_-$$

WEDGE INCLUSION

$$g_1 W \subset W \quad :\Longleftrightarrow \quad g_1 \in S_W$$

ALL THE INCLUSIONS ARE GENERATED BY

HALF-SIDED INCLUSIONS



$$\pm\mathrm{HSI}: \quad e^{x_\pm t} W \subset W \qquad x_\pm \in C_\pm, \; t \geq 0$$

CHOICE OF POSITIVE CONE $\Rightarrow$ POSET OF WEDGES

HAAG-KASTLER NET

LET $\mathcal{W}_+$ BE A $G^\uparrow$-TRANSITIVE LOCAL POSET OF WEDGES W.R.T. A CONE C

LET U BE UNITARY REPRESENTATION OF $G^\uparrow$ ON $\mathcal{H}$

$$\mathcal{W}_+ \ni W \longrightarrow H(W) \in \mathcal{H}$$

* ISOTONY $\quad W_1 \subset W_2 \Rightarrow H(W_1) \subset H(W_2)$
* LOCALITY $\quad W_1 \subset W_2' \Rightarrow H(W_1) \subset H(W_2)'$
* $G^\uparrow$-COVARIANCE $\quad U(g)\, H(W) = H(g.W) \qquad g \in G^\uparrow$
* POSITIVITY OF THE ENERGY $\quad C \subseteq C_U = \{ x \in \mathfrak{g} \mid -i\, \partial U(x) \geq 0 \}$
* CYCLICITY $\quad H(W)$ STANDARD SUBSPACES
* BW-PROPERTY $\quad U(e^{x_W t}) = \Delta_{H(W)}^{-\frac{it}{2\pi}} \qquad \left(\Rightarrow H(W') = H(W)' \right)$
* PCT THEOREM $\quad J_W$ extends U to a G COVARIANT REPRESENTATION

$$U(\sigma_W) := J_{H(W)}$$

MODULAR CONSTRUCTION

U (ANTI-)UNITARY POSITIVE ENERGY REPRESENTATION OF $G = G^{\uparrow} \cup G^{\downarrow}$

$$W_+ \ni W \longmapsto H(W = (x_W, \sigma_W)) \equiv \left(e^{-2\pi\, \partial U(x_W)},\ U(\sigma_W) \right)$$

BGL CONSTRUCTION IS POSSIBLE AND SATISFY ALL THE ASSUMPTIONS

IN PARTICULAR:

SPECTRAL CONDITION
$G^{\uparrow}$ COVARIANCE
B-W PROPERTY

$\Rightarrow$ ISOTONY

Eg $G^{\uparrow} = PSL_2(\mathbb{R})$

$*$ $C = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl_2(\mathbb{R}) : b \geq 0,\ c \leq 0,\ a^2 \leq -bc \right\} \left(\supseteq \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} : t \in \mathbb{R} \right\} \right)$

SEVERAL WEDGE INCLUSIONS -

- DISCRETE SERIES REPRESENTATIONS HAVE POSITIVITY OF THE ENERGY (P.E.)

$*$ $C = \{0\}$, NO WEDGE INCLUSION $\Rightarrow$ NET ON THE CIRCLE

- PRINCIPAL/COMPLEMENTARY SERIES REPRESENTATIONS HAVE NO POSITIVE ENERGY

$\Rightarrow$ NET ON DE SITTER (see also Guido-Longo 2003)

LOOKING FOR NEW MODELS

LET $\mathfrak{g}$ BE A NON-COMPACT REAL SIMPLE LIE ALGEBRA,

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ CARTAN DECOMPOSITION,

$\mathfrak{a} \subset \mathfrak{p}$ CARTAN SUBALGEBRA $\Rightarrow$ STUDY OF THE RESTRICTED ROOT SPACE TO LOOK FOR WEDGE ORBITS

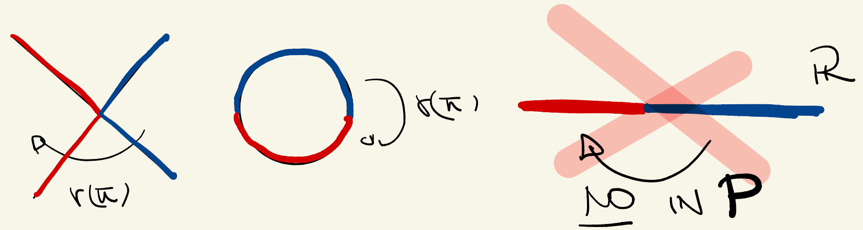
HERE THE ORBITS ARE CONSIDERED W.R.T THE $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathfrak{g}} \rangle$ ACTION ON $\mathfrak{g}$

Thm NUMBER OF ORBITS OF EULER ELEMENTS

Pt 1

A_n : n-ORBITS	B_n : 1-ORBIT	C_n : 1 ORBIT
D_n : 3 ORBITS	E_6 : 2-ORBITS	E_7 : 1 ORBIT

▷ W is symmetric if $W' = g.W$, $g \in G^{\uparrow}$



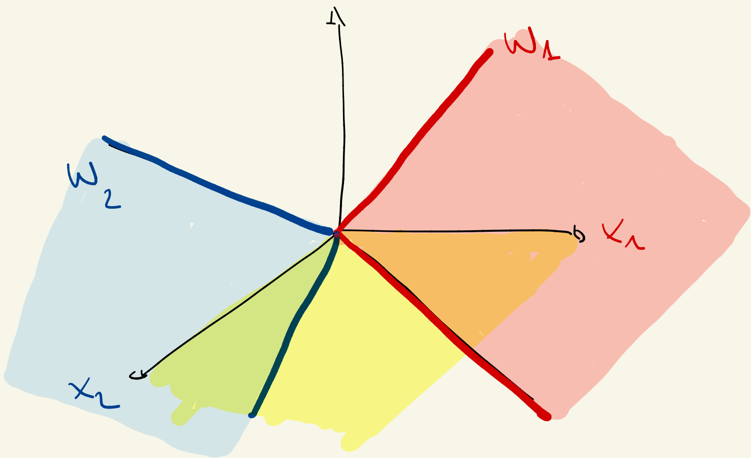
Thm NUMBER OF ORBITS OF SYMMETRIC EULER ELEMENTS

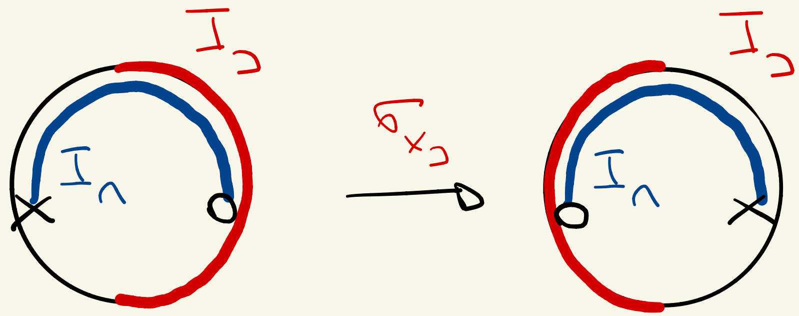
Pt 2

A_{2n-1} : 1-ORBIT	B_n : 1-ORBIT	C_n : 1-ORBIT
D_n : 1-ORBIT	D_{2n} : 2-ORBITS	E_7 : 1-ORBIT

About The Net Structure

Def Orthogonal Euler elements (x, y): $\boxed{\sigma_x(y) = -y}$





$W_i = \{x \in \mathbb{R}^{1+s} : |t| < x_i\},\ i = 1, 2.$

Thm $x, y \in \mathfrak{g}$ Euler orthogonal elements (x, y)

* x, y are symmetric
* $\mathrm{Lie}(x, y) = \mathfrak{sl}_2(\mathbb{R})$
* (y, x) orthogonal elements (symmetric property)

COMMENTS

▷ IN CASE: $Z(G) \neq \{e\}$,

- * SEVERAL WEDGE ORBITS
- * SEVERAL WEDGE COMPLEMENT PARAMETRIZED BY CENTRAL ELEMENT
- * TWISTED LOCALITY RELATION

$\Rightarrow$ EVERYTHING COMPLIES WITH THE H-K PICTURE (WITH MODIFICATIONS) $\Rightarrow$ OK WITH MODELS ON S^1 COVERINGS OR FERMIONIC NETS

OUTLOOK

▷ POSSIBLE NEW MODELS ON ALGEBRAIC GROUPS

▷ POSSIBLE DEFINITION OF SPACELIKE CONES

THANK YOU FOR YOUR ATTENTION !