Entropy in Causal Fermion Systems

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Preface

- Joint work with Felix Finster and Robert Jonsson
- Theory of causal fermion systems [3, p. 1]:
  - Novel approach to fundamental physics
  - Unification of interactions of the standard model with gravity
  - Close connections to quantum field theory
- Why consider entropy?
  - Plays important role in many physical effects
  - Many different definitions
  - Correlations not always well understood (e.g. area law)
  - Part of problems like black hole information paradox (see also [5] and [7])

Outline

Causal Fermion Systems [3, p. 1, 3]

Definition (Causal Fermion System)

A *causal fermion system* (short CFS) of spin dimension $n \in \mathbb{N}$ ia a triple $(\mathcal{H}, \mathcal{F}, \rho)$ consisting of a complex separable Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$, the following subset of $L(\mathcal{H})$:

$$\mathcal{F} := \{x \in L(\mathcal{H}) \mid x \text{ selfajoint with } \mathrm{rank}(x) \leq 2\mathrm{n} \text{ and at most } n \text{ positive} \\ \text{and at most } n \text{ negative eigenvalues}\} ,$$

and a measure ρ on $\mathcal{F}$, the so called *universal measure*.

Definition (Spacetime)

For a CFS $(\mathcal{H}, \mathcal{F}, \rho)$ we define *spacetime* as:

$$M := \mathrm{supp}(\rho) \subseteq \mathcal{F} \subseteq L(\mathcal{H}) .$$

Causal Action Principle [3, p. 1-4]

Lagrangian: $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}\,, \quad (x, y) \mapsto \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \,,$

with λ_i^{xy}, $i = 1, \ldots, 2n$ (non-zero) eigenvalues of xy

- non-negative
- symmetric

Definition (Causal Action Principle)

Minimize the action

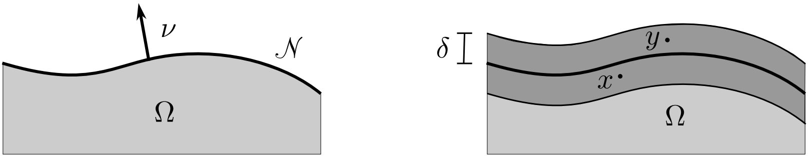
$$S(\rho) := \int_{\mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y)\,,$$

by varying ρ under the constraints

- *Volume constraint:* $\rho(\mathcal{F})$ constant,
- *Trace constraint:* $\int_{\mathcal{F}} \operatorname{tr}(x) d\rho(x)$ constant,
- *Boundedness constraint:* $\int_{\mathcal{F}} \int_{\mathcal{F}} \left(\sum_{j=1}^{2n} |\lambda_i^{xy}| \right)^2 d\rho(x) d\rho(y) \le C$,

- From now on we always assume that ρ is such a minimizer

Surface Layer Integrals [3, p. 42-43]



$$\int_{\mathcal{N}} \cdots \, d\mu_{\mathcal{N}} \qquad \int_{\Omega} d\rho(x) \int_{M\setminus\Omega} d\rho(y) \, \cdots \, \mathcal{L}(x,y)$$

Figure: Comparison of an ordinary surface integral and a surface layer integral [3, p. 42].

- In CFS no canonical measure on hypersurface
- $\mathcal{L}$ decays fast for $|x-y| > \delta$ (for Dirac particles: $\delta \sim m^{-1}$)

$\rightarrow$ gives integral over hypersurface smeared out on the scale of δ

A conserved surface layer integral [3, S. 42-45], [4]

- For $u \in \mathcal{H}$ arbitrary set $\mathcal{A}_u := |u\rangle \langle u| \in L(\mathcal{H})$
  - symmetric
  - finite rank

$\rightarrow$ One-parameter family of unitary transformations: $(U_\tau)_{\tau \in \mathbb{R}} := (\exp(i\tau \mathcal{A}_u))_{\tau \in \mathbb{R}}$

$\rightarrow$ Symmetry $\phi : \mathbb{R} \times \mathcal{F} \rightarrow \mathcal{F}\,,\ (\tau, x) \mapsto U_\tau x U_\tau^{-1}$ of $\mathcal{L}$, i.e.:

$$\mathcal{L}(\phi(\tau, x), y) = \mathcal{L}(x, \phi(-\tau, y))\,, \quad \forall \tau \in \mathbb{R}, x, y \in M\,.$$

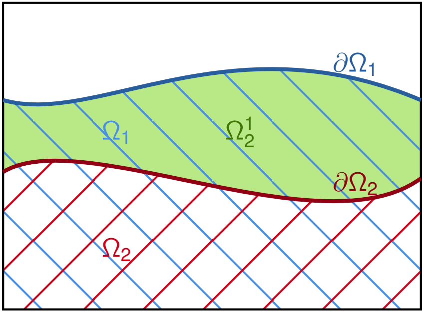
Lemma

For any $\Omega \subseteq M$ compact:

$$0 = \frac{d}{d\tau} \left(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \Big(\mathcal{L}(\phi(\tau, x), y) - \mathcal{L}(x, \phi(\tau, y)) \Big) \right) \Big|_{\tau=0}$$
$$= \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \Big(D_{1, v_u(x)} - D_{2, v_u(y)} \Big) \mathcal{L}(x, y)\,,$$

with $v_u(x) := \frac{d}{d\tau}\phi(\tau, x)|_{\tau=0}$ for all $x \in \mathcal{F}$.

- $\Omega_{1/2}$: past of Cauchy surface $\partial\Omega_{1/2}$
- Ω_1 in the future of Ω_2
- Meantime $\Omega_2^1 := \Omega_1 \setminus \Omega_2$ can be exhausted by compact sets $(\tilde{\Omega}_n)_{n\in\mathbb{N}}$



$\Rightarrow$ For the integrand decaying fast enough at spatial infinity:

$$0 = \int_{\Omega_2^1} d\rho(x) \int_{M\setminus\Omega_2^1} d\rho(y)\Big(D_{1,v_u(x)} - D_{2,v_u(y)}\Big)\mathcal{L}(x,y)$$

$$= \Big(\int_{\Omega_1}\int_{M\setminus\Omega_1} - \int_{\Omega_2}\int_{M\setminus\Omega_2}\Big)\Big(D_{1,v_u(x)} - D_{2,v_u(y)}\Big)\mathcal{L}(x,y)d\rho(y)d\rho(x)$$

$\rightarrow$ $\langle u,u\rangle_\rho := \int_\Omega d\rho(x)\int_{M\setminus\Omega} d\rho(y)\Big(D_{1,v_u(x)} - D_{2,v_u(y)}\Big)\mathcal{L}(x,y)$ conserved for Ω past of Cauchy surface $\partial\Omega$

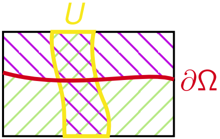
Definition of the Entropy

- In Minkowski space limiting case (see [4, Chapter 5]):

$$\langle u, u\rangle_\rho = c\langle u|u\rangle_{\mathcal{H}} \quad \forall u \in \mathcal{H}_F \subseteq \mathcal{H} \text{ closed subspace} .$$

→ Assume this holds in general for $\mathcal{H}_F$ suitable

→ $\langle .,.\rangle_\rho$ inner product on $\mathcal{H}_F$



- Spatial localization for $U \subseteq M$:

$$\underbrace{\frac{1}{2}\Big(\int_{\Omega\cap U}\int_{M\setminus\Omega} + \int_{\Omega}\int_{(M\setminus\Omega)\cap U}\Big)\Big(D_{1,v_u(x)} - D_{2,v_u(y)}\Big)\mathcal{L}(x,y)d\rho(x)d\rho(y)}_{=:\langle u|u\rangle_{U,\rho}}$$

- Define localized state operator

$$\langle u|u\rangle_{U,\rho} = \langle u, \sigma_U u\rangle_\rho \,, \quad \forall u \in \mathcal{H}_F$$

- Von-Neumann-like entropy (for reduced one-particle density) [6, (34)], [8, p. 400-401]

$$S := \mathrm{tr}_{\mathcal{H}_F}\big(\sigma_U \log(\sigma_U) + (1-\sigma_U)\log(1-\sigma_U)\big)$$

A more specific Form of σ_U [3, p. 39-40], [4, chapter 5]

- Using a kernel Q the product $\langle ., . \rangle_\rho$ can be rewritten as:

$$\langle u|u\rangle_\rho = 4\mathrm{Im} \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \prec \psi^u(x), Q(x,y)\psi^u(y) \succ_y$$

- And correspondingly:

$$\langle u|u\rangle_{U,\rho} = 2\mathrm{Im}\Big(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) \prec \psi^u(x), Q(x,y)\chi_U(y)\psi^u(y) \succ_y + \prec \chi_U(x)\psi^u(x), Q(x,y)\psi^u(y) \succ_y \Big)$$

$\rightarrow$ More specific form for σ_U:

$$\sigma_U = \frac{1}{2}((\chi_U)^* + \chi_U) \,,$$

with $(\chi_U)^*$ adjoint of χ_U wrt $\langle .|. \rangle_\rho$

- In vakuum Mionkowski space: $\sigma_U = \sigma \tilde{\sigma}_U \sigma$ with
  - $\sigma : L^2(\mathbb{R}^3, \mathbb{C}^4) \to \mathcal{H}_F$ projection operator with kernel $\sigma(\vec{x}, \vec{y})$ s.t.:

$$\sigma(\vec{x}, \vec{z}) = \int_{\mathbb{R}^3} d^3\vec{y} \sigma(\vec{x}, \vec{y}) \sigma(\vec{y}, \vec{z}) \ .$$

  - $\tilde{\sigma}_U : L^2(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ integral operator with kernel

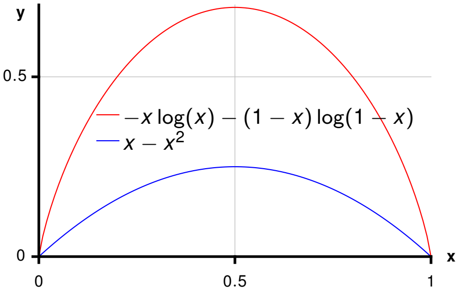
$$\tilde{\sigma}_U(\vec{x}, \vec{y}) = \frac{1}{2}\big(\chi_U(\vec{x}) + \chi_U(\vec{y})\big)\sigma(\vec{x}, \vec{y}) \ .$$

$\Rightarrow$ Assume in the follwoing that σ_U can always be represented in this form

Consider $\sigma_U - (\sigma_U)^2$

- $\log(\sigma_U)$ difficult to calculate
- Why log-formula at all?
- $x - x^2$ similar behaviour:
  - EVs 0, 1 don't contribute
  - → Measues mixing

  plus easier to calculate (similar as in [2, Theorem 3.1]):



$$\mathrm{tr}(\sigma_U - (\sigma_U)^2) = \mathrm{tr}(\sigma_U - \sigma_U(\sigma - \sigma_{\mathbb{R}^3\setminus U})) = \mathrm{tr}(\sigma_U \sigma_{\mathbb{R}^3\setminus U}) = \mathrm{tr}(\tilde{\sigma}_U \sigma \tilde{\sigma}_{\mathbb{R}^3\setminus U}\sigma)$$

$$= \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\chi_U(\vec{x}_1) + \chi_U(\vec{x}_2)\Big)\Big(\chi_{\mathbb{R}^3\setminus U}(\vec{x}_3) + \chi_{\mathbb{R}^3\setminus U}(\vec{x}_4)\Big)\cdot$$

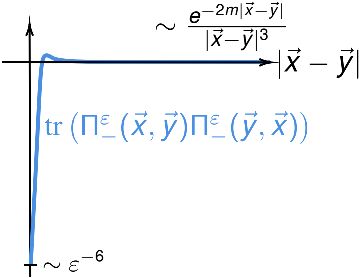
$$\mathrm{tr}_{\mathbb{C}^4}\Big(\sigma(\vec{x}_1,\vec{x}_2)\sigma(\vec{x}_2,\vec{x}_3)\sigma(\vec{x}_3,\vec{x}_4)\sigma(\vec{x}_4,\vec{x}_1)\Big)\, d^3\vec{x}_1\, d^3\vec{x}_2\, d^3\vec{x}_3\, d^3\vec{x}_4$$

$$= \int_U d^3\vec{x}\int_{\mathbb{R}^3\setminus U} d^3\vec{y}\ \mathrm{tr}_{\mathbb{C}^4}\Big(\sigma(\vec{x},\vec{y})\sigma(\vec{y},\vec{x})\Big)\,,$$

Minkowski Space Example (see also [3, p. 25-34])

- Vacuum Minkowski space: $\sigma = \Pi_-^\varepsilon$ regularized projector to negative frequency solutions of Dirac equation

- Important properties of $\mathrm{tr}(\Pi_-^\varepsilon(\vec{x},\vec{y})\Pi_-^\varepsilon(\vec{y},\vec{x}))$:
  - only depends only on $|\vec{x}-\vec{y}|$
  - diverges for $|\vec{x}-\vec{y}| \to 0$ and $\varepsilon \to 0$
  - decays fast for large $|\vec{x}-\vec{y}|$



$\to$ Split surface layer integral with cutoff constant $0 < \varepsilon \ll \delta \ll l$:

$$\int_{B_l(0)} \int_{\mathbb{R}^3 \setminus B_l(0)} \cdots = \int_{B_l(0)} \int_{B_l(0)^c \cap B_\delta(\vec{x})} \cdots + \int_{B_l(0)} \int_{B_l(0)^c \cap B_\delta(\vec{x})^c} \cdots$$

- In the limit $\varepsilon \to 0$ (physical case) fist term diverges, second one bounded $\to$ focus on first term

- Simplifying integrals, the first term becomes:

$$\int_{B_l(0)} d^3\vec{x} \int_{B_l(0)^c \cap B_\delta(\vec{x})} d^3\vec{y}\,\mathrm{tr}_{\mathbb{C}^4}(\Pi_-^\varepsilon(\vec{x},\vec{y})\Pi_-^\varepsilon(\vec{y},\vec{x}))$$
$$= C \cdot \left[l^2 \int_0^\delta dr\, r^3 F_\varepsilon(r^2) - \frac{1}{12} \int_0^\delta dr\, r^5 F_\varepsilon(r^2) \right],$$

with $F_\varepsilon : \mathbb{R} \to \mathbb{C}$ such that $F_\varepsilon(r^2) = \mathrm{tr}_{\mathbb{C}^4}(\Pi_-^\varepsilon(0, r\hat{e}_1)\Pi_-^\varepsilon(0, r\hat{e}_1))$ for all $r \geq 0$

- First integral dominates for $l \gg \delta$

→ obtain area-law-like form:

$$\mathrm{tr}(\sigma_{B_l(0)} - \sigma^2_{B_l(0)})$$
$$= \underbrace{C(\varepsilon)}_{\xrightarrow[\varepsilon \to 0]{} \infty} \cdot \Big(C_0(\delta) \cdot l^2 + \underbrace{C_1(\delta)}_{\ll C_0(\delta) l^2 \text{ for } l \gg \delta} \Big) + \underbrace{D(\varepsilon)}_{\mathcal{O}(1) \text{ for } \varepsilon \to 0} \cdot \text{Corrections},$$

Generalization to the logarithm

- Taylor expansion of logarithm:

$$\mathrm{tr}\Big((1-\sigma_U)\ln(1-\sigma_U)\Big) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{tr}(\sigma_U^n) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{tr}(\sigma_U^{n+1}) \,,$$

→ Additivity:

$$\mathrm{tr}(\sigma_U^{n+1}) = \mathrm{tr}((\sigma_U)^n(\sigma - \sigma_{\mathbb{R}^3\setminus U})) = \mathrm{tr}((\sigma_U)^n\sigma) - \mathrm{tr}((\sigma_U)^n\sigma_{\mathbb{R}^3\setminus U}) \,,$$

→ By definition of σ_U: $\mathrm{tr}((\sigma_U)^n\sigma) = \mathrm{tr}((\sigma_U)^n)$

→ And similar as before: $\mathrm{tr}((\sigma_U)^n\sigma_{\mathbb{R}^3\setminus U}) = \mathrm{tr}((\chi_U\sigma)^n(\chi_{\mathbb{R}^3\setminus U}\sigma))$

→ Finially iteratively use:

$$\mathrm{tr}\Big((\chi_U\sigma)^n(\chi_{\mathbb{R}^3\setminus U}\sigma)\Big) = \mathrm{tr}\Big((\chi_U\sigma)^{n-2}(1-\chi_{\mathbb{R}^3\setminus U})\sigma(\chi_U\sigma)(\chi_{\mathbb{R}^3\setminus U}\sigma)\Big)$$

$$= \mathrm{tr}\Big((\chi_U\sigma)^{n-1}(\chi_{\mathbb{R}^3\setminus U}\sigma)\Big) - \mathrm{tr}\Big((\chi_U\sigma)^{n-2}(\chi_{\mathbb{R}^3\setminus U}\sigma)(\chi_U\sigma)(\chi_{\mathbb{R}^3\setminus U}\sigma)\Big) \,,$$

- Proceed similarly with $\mathrm{tr}(\sigma_U \log(\sigma_U)) = \mathrm{tr}(\sigma_U \log\big(1 - \sigma_{\mathbb{R}^3 \setminus U}\big))$

$\rightarrow$ After simplification this yields expansion:

$$\mathrm{tr}\Big(\sigma_U \log(\sigma_U) + (1-\sigma_U)\ln(1-\sigma_U)\Big) = \sum_{n=1}^{\infty} \alpha_n \mathrm{tr}\Big(\big((\chi_U \sigma)(\chi_{\mathbb{R}^3 \setminus U} \sigma)\big)^n\Big) .$$

- Resulting traces correspond to higher order surface layer integrals (alternating integrals over inside and outside)
- Conjecture:
  - Short range of integrand $\rightarrow$ for each pair of integrals over U and U^c only small strips of space around ∂U relevant
  - Surface layer integrals decay with growing order
  - Only lower orders in n relevant
  - Lowest order: Ordinary surface layer integral, which likely has area-law-like form

Outlook

- Prove more general area-law-like form for area A_U of U:

$$S = \underbrace{C(\varepsilon)}_{\substack{\longrightarrow \infty \\ \varepsilon \to 0}} \cdot \left(C_0(\delta) \cdot A_U + \underbrace{C_1(\delta)}_{\ll C_0(\delta) A_U \text{ for } A_U \gg \delta^2} \right) + \underbrace{D(\varepsilon)}_{\mathcal{O}(1) \text{ for } \varepsilon \to 0} \cdot \text{Corrections}$$

- Show that $S := \mathrm{tr}\big(\sigma_U \log(\sigma_U) + (1 - \sigma_U) \log(1 - \sigma_U)\big)$ really corresponds to entanglement entropy

$\rightarrow$ Consequences for information paradox: General problem [5], [7]:
  - Independent of the initial state a black hole evaporates by emitting Hawking radiation
  - Shrinking horizon $\rightarrow$ Bekenstein-Hawking entropy shrinks
  - Thermal radiation $\rightarrow$ Entanglement-entropy grows
  - Problem if Bekenstein-Hawking entopy and entanglement-entropy are always equal

- Hope: Answer question whether they really are equal all the time

[1]: CFS-website: https://causal-fermion-system.com/

[2]: Chris Brislawn: *Traceable Integral Kernels on Countably Generated Measure Spaces*, Pacific Journal of Mathematics, **150** (1991), no. 2.

[3]: Felix Finster: *The Continuum Limit of Causal Fermion Systems*, arXiv:1605.04742 [math-ph], Fundamental Theories of Physics, vol. 186, Springer, 2016, (version 3, 2018).

[4]: Felix Finster, Johannes Kleiner: *Noether-like theorems for causal variational principles*, arXiv:1506.09076 [math-ph], Calculus of Variations and Partial Differential Equations, **55** (2016), no. 2, (version 2, 2016).

[5]: Daniel Harlow: *Jerusalem Lectures on Black Holes and Quantum Information*, arXiv:1409.1231 [hep-th], Rev. Mod. Phys., **88** (2016), no. 1 (version 4, 2015).

[6]: Robert Helling, Hajo Leschke, Wolfgang Spitzer: *A Special Case of a Conjecture by Widom with Implications to Fermionic Entanglement Entropy*, arXiv:0906.4946 [math-ph], International Mathematics Research Notices, 2010, (version 2, 2010).

[7]: Leonard Susskind, James Lindesay, *An Introduction To Black Holes, Information And The String Theory Revolution: The Holographic Universe*, World Scientific Publishing Company, Singapur, 2005.

[8]: Walter Thirring: *Quantum Mathematical Physics: Atoms, Molecules and Large Systems*, Springer, Berlin, Heidelberg 2002 (second edition, second printing 2003).